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Wave Theory of Impact and Professor Yury Rossikhin Contribution in the Field (A Memorial Survey)

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This survey article overviews the impact response of solids and structures within the framework of the wave theory of impact. It is dedicated to the bright memory of Professor Yury A. Rossikhin who has contributed a lot in the development of the wave theory of impact based on the theory of discontinuities and ray expansions, resulting in analytical solutions of intricate problems of impact interaction of solids possessing different features.

Keywords conditions of compatibility, fractional calculus models, Hertz's contact law, ray expansions, theory of discontinuities, viscoelastic impact, wave theory of impact

1. Introduction

The problems connected with the analysis of the shock interaction of thin bodies (rods, beams, plates, and shells) with other bodies have widespread application in various fields of science and technology. The physical phenomena involved in the impact event include structural responses, contact effects, and wave propagation (Ref 1). These problems are topical not only from the point of view of fundamental research in applied mechanics, but also with respect to their applications. Since these problems belong to the problems of dynamic contact interaction, their solution is connected with severe mathematical and calculation difficulties. To overcome this impediment, a rich variety of approaches and methods have been suggested (Ref 1-5), among them the approaches based on the generation of shock waves (surfaces of strong discontinuity) at the moment of impact and their further propagation along the bodies are of greatest practical use (Ref 1, 6-8). These approaches use hyperbolic sets of equations to describe the dynamic behavior of thin bodies and are called *wave approaches* (Ref 1). As for the non-wave approaches, which use classical equations to describe the dynamic behavior of thin bodies, then Zener's approach (Ref 9), in which a thin body is assumed to be large enough in order to ignore stable waves, is closely related to the wave approach. Both non-wave and wave approaches are finally reduced to one linear or nonlinear differential or integro-differential equation in some characteristic value, for example, the indentation, or the contact force, or the thin body's displacement at the place of contact.

Usually, it is possible to obtain the solution of such a nonlinear equation in the form of a power series in time with integer and/or fractional powers, which suggests that the main functions of the shock interaction could be represented in terms of power series both inside and outside the contact domain.

Such an assumption has been verified in approaches proposed by Professor Yury A. Rossikhin (which have been overviewed in detail in Ref 1, 10), where the ray series (Ref 6-8) that involve as a variable the difference between the current time and the time of the wave arrival at the given point are used as power expansions outside the contact zone. On the contact domain boundary, the ray series go over into the power series with respect to time, while those characteristic values that have a non-wave characteristic and describe the impact process inside the contact domain are represented in terms of the power series in time with unknown coefficients to be determined.

The wave theory of impact based on the ray method was developed first for elastic bodies such as beams, plates, shells (Ref 11-23), considering different types of materials: isotropic (Ref 11-17), anisotropic (Ref 21, 22), prestressed (Ref 18-21), and composites possessing the Cosserat-type microstructure (Ref 23).

Further it has been generalized for thermoelastic bodies (Ref 24-33) considering two models of hyperbolic thermoelasticity: the Lord-Shulman approach (Ref 34) with thermal relaxation (Ref 31-33) and thermoelasticty without energy dissipation (Ref 24-30), which was proposed by Rossikhin in 1978 (Ref 24) and Green and Nagdy in 1993 (Ref 35).

Another extension of the wave theory of impact, which is very important for engineering applications, was carried out for viscoelastic bodies (Ref 36-58), Young's moduli of which are time-dependent operators defined either via the Kelvin–Voigt fractional derivative model or via the standard linear solid fractional derivative model (the review of the fractional calculus models and their application in dynamic problems of mechanics of solids and structures could be found in Ref 59-61).

This survey is dedicated to the bright memory of Professor Yury A. Rossikhin (1944–2017), who was a Distinguished Researcher of the Russian Federation, a Highly Cited Researcher due to Web of Science records, and a scientist with the encyclopedical knowledge in Dynamics of Solids and Structures, involving wave theory of impact.

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Professor Yury A. Rossikhin
(14.03.1944 – 29.03.2017)

2. Modeling the Elastic Impact Response of Thin Bodies Based on the Hertzian Contact Law or Its Modifications

The problems dealing with the impact response of elastic thin bodies accompanied by transient wave propagation have been overviewed in the state-of-the-art article by Rossikhin and Shitikova (Ref 1), which was published in 2007 in *The Shock and Vibration Digest*, wherein all the papers that are of interest in our consideration have been classified by the characteristic of the interaction of thin bodies with other bodies within the contact domain, as well as by the characteristic behavior of the thin bodies outside the contact region. In turn, the types of interaction and behavior are dictated by the dimensions of the thin bodies and the shape of an impactor and are governed by the set of equations that describe their dynamic response, by the type of contact force, and so on.

In Ref 1, thin bodies of infinite extent and of finite dimensions have been considered, the dynamic behavior of which is described by both classical and non-classical systems of equations. Thus, the contact force may be linear or nonlinear, and it may be dependent on displacements or on velocities of displacements, or both. Impactors may possess the shape of a sphere, a short cylinder, a long rod, etc.

In the present paper, we will first mention briefly the backgrounds of the elastic impact approach and provide the main idea of the ray expansions in the wave theory of impact in order to focus then on the results obtained in the field during last decade.

2.1 The Hertz Contact Law

It is well known that in the case when the impactor's initial velocity is rather low, a transverse shear wave propagates along a thin body (a) either with an infinitely large velocity (Bernoulli–Euler beam, Kirchhoff–Love plate) or (b) with a finite velocity (Timoshenko beam, Uflyand–Mindlin plate). The motions of impacted bodies outside the contact domain are described usually by equations of vibrations generated by the action of a concentrated contact force.

During the loading phase, the contact force P is related to the indentation α

$$P = k\alpha^n \qquad \text{(Eq 1)}$$

until the maximum indentation $\alpha_{\max}$ and the corresponding maximum force $P_{\max}$ are reached. According to the Hertzian theory of contact (Ref 62) $n = 3/2$

$$P = k\alpha^{3/2} \qquad \text{(Eq 2)}$$

where

$$k = \frac{4}{3}E^*\sqrt{R} = \frac{4\sqrt{R}}{3\pi(k_{\rm im} + k_t)} \qquad \text{(Eq 3)}$$

$$\frac{1}{R} = \frac{1}{R_{\rm im}} + \frac{1}{R_t} \qquad \text{(Eq 4)}$$

$$\frac{1}{E^*} = \frac{1-\sigma_{\rm im}^2}{E_{\rm im}} + \frac{1-\sigma_t^2}{E_t} = \pi(k_{\rm im} + k_t) \qquad \text{(Eq 5)}$$

Here $R_{\rm im}$, $E_{\rm im}$, $\sigma_{\rm im}$ and R_t, E_t, $\sigma_{\rm t}$ are the radius, Young's modulus and Poisson's ratio of the impactor and target, respectively. In this case, the radius of the contact zone a is connected with the relative displacement α as

$$a = R^{1/2}\alpha^{1/2} \qquad \text{(Eq 6)}$$

To account for the permanent deformation and to model the unloading phase, Crook (Ref 63) proposed the equation

$$P = P_{\max}\left(\frac{\alpha - \alpha_{cr}}{\alpha_{\max} - \alpha_{cr}}\right)^q, \quad \alpha_{\max} \ge \alpha \ge \alpha_{cr} \qquad \text{(Eq 7)}$$

where α_{cr} is the permanent crater depth, and q = 1.5–2.5 is the material constant.

2.2 Impact upon Beams and Plates of Finite Dimensions

Timoshenko (Ref 64) considered the problem on a transverse impact of an elastic sphere upon an elastic Bernoulli–Euler beam. In this problem, the equations of motion of a projectile and a beam have the form

$$m\ddot{y} = -P(t) \qquad \text{(Eq 8)}$$

$$EI\frac{\partial^4 w}{\partial x^4} + \rho F\ddot{w} = P(t)\delta(x - \xi) \qquad \text{(Eq 9)}$$

with the initial conditions

$$y(0) = 0, \quad \dot{y}(0) = V_0, \quad w(x,0) = 0, \quad \dot{w}(x,0) = 0 \qquad \text{(Eq 10)}$$

where w is the beam's deflection at the point of contact, EI is the beam's bending rigidity, I and F are the moment of inertia and cross-sectional area, respectively, V_0 is the initial velocity of impact, $\delta(x - \xi)$ is the Dirac delta-function which positions the point ξ of the contact force application, a dot denotes the time-derivative, m is the mass of the sphere, y is the displacement of the sphere

$$y = \alpha + w(\xi, t) \qquad \text{(Eq 11)}$$

and local bearing α of the impactor's and target's materials at the contact point

$$\alpha = k^{-3/2} P^{2/3} \quad \text{(Eq 12)}$$

Integrating equations (Eq 8) and (Eq 9) due with account for the initial conditions (Eq 10), we could obtain the functional equation either for determining the contact force $P(t)$

$$V_0 t - \frac{1}{m}\int_0^t \int_0^{t_1} P(t_2) dt_2 dt_1 = k' P^{2/3} + \sum_n A_n \int_o^t P(\tau) \sin\omega_n (t-\tau) d\tau \quad \text{(Eq 13)}$$

or for the indentation α

$$\ddot{\alpha} + k\left(\frac{1}{m} + \sum_{(n)} \omega_n A_n\right)\alpha^{3/2} - k\int_0^t \alpha^{3/2}(\tau) \sum_{(n)} \omega_n^2 A_n \sin\omega_n (t-\tau) d\tau = 0 \quad \text{(Eq 14)}$$

where the coefficients A_n depend on the eigenfunctions and eigenvalues ω_n of the problem under consideration. Equations (Eq 13) and (Eq 14) could be solved numerically (Ref 2).

The basic approach by Timoshenko (Ref 64) was used by Karas (Ref 65) for analyzing the impact response of a rectangular simply supported Kirchhoff–Love plate. A lot of references to works using this approach could be found in Ref 1.

2.3 Impact upon Infinitely Extended Thin Bodies

Zener (Ref 9) was a pioneer in solving the problem of the elastic normal impact of spheres with infinitely large Kirchhoff–Love plates. Because of its importance, this problem can be placed along with the Timoshenko problem (Ref 64).

Infinite dimensions of the plate allowed Zener (Ref 9) to obtain the relationship between the plate's mid-plane displacement directly underneath the point of load application and the impulse of the contact force:

$$w(0,t) = \beta \int_0^t P(t) dt \quad \text{(Eq 15)}$$

or

$$\dot{w}(0,t) = \beta P(t) \quad \text{(Eq 16)}$$

where $\beta = \frac{1}{8}(D\rho h)^{-1/2}$, and D is the cylindrical rigidity, resulting in

$$\ddot{\alpha} + \frac{3}{2} b \alpha^{1/2} \dot{\alpha} = -\frac{k}{m}\alpha^{3/2} \quad \text{(Eq 17)}$$

where $b = \beta\ k$.

Putting $A = \dot{\alpha}$ and going from the independent variable t to the new independent variable α, we are led to the equation (Ref 1)

$$A\frac{dA}{d\alpha} + \frac{3}{2} b \alpha^{1/2} A = -\frac{k}{m}\alpha^{3/2} \quad \text{(Eq 18)}$$

subjected to the initial condition $A|_{\alpha=0} = V_0$.

The analytical solution of this nonlinear differential equation was presented in Ref 1 in the following form:

$$A = V_0 + \sum_{i=1}^{\infty} a_i \alpha^{(2i+1)/2} + \sum_{i=1}^{\infty} b_i \alpha^i \quad \text{(Eq 19)}$$

Substituting (Eq 19) into equation (Eq 18) and equating the coefficients at integer and fractional powers of α, we are led to the set of two recurrent equations for defining the coefficients a_i and b_i.

Zener's approach became very popular, and a huge amount of problems dealing with impact response of plates and shells including with anisotropic and composite materials have been solved (see references in Ref 1).

2.4 Combining the Hertzian Contact Theory with the Wave Approach

In the case when dynamic behavior of the target is described by the so-called Timoshenko-type theories considering the rotary inertia and transverse shear deformations (a Timoshenko beam, an Uflyand–Mindlin plate or Reissner-Ambatsuymyan shell, or other refined theories of higher order), the local indentation during contact interaction of bodies could be analyzed according the Hertzian contact theory or its modifications as well; however, the dynamic deformation of the target's material outside the contact domain now is caused by the propagating transient wave of transverse shear generated at the moment of impact. Thus, the approaches discussed in previous section 2.1 are inapplicable in such cases. The wave approach could be in use to solve the problems of such a kind.

For this purpose, Professor Rossikhin suggested to utilize the theory of discontinuities based on the ray expansions (Ref 8), according to which the solution behind the front of the wave of the strong discontinuity Σ is constructed in terms of the ray series (Ref 6)

$$Z(s,t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[Z_{,(k)}\right]_{t=s/G} \left(t - \frac{s}{G}\right)^k H\left(t - \frac{s}{G}\right) \quad \text{(Eq 20)}$$

where Z is the desired function, $Z_{,(k)} = \partial^k Z / \partial t^k$, $\left[Z_{,(k)}\right] = Z^{+}_{,(k)} - Z^{-}_{,(k)}$, the signs "+" and "–" refer to the magnitudes of the derivative $Z_{,(k)}$ calculated before and behind the wave surface Σ, respectively, G is the normal velocity of the wave Σ, $H(t - s/G)$ is the unit Heaviside function, s is the arc length calculated along the ray, and t is the time.

If the duration of the process is small, then one could be restricted to the first term of the ray series (Eq 20) only. In this case, the problem is reduced to the solution of one nonlinear differential equation with respect to the value characterizing the local indentation or with respect to the contact force. The solution of such an equation is either constructed in terms of a power series with fractional exponents or found numerically.

In order to refine the one-term solution, succeeding terms of the ray series could be taken into account. For this purpose, one should differentiate the governing equations for the contacting body k times with respect to time, write them on the different sides of the wave surface, and take their difference. Then the conditions of compatibility should be used, which in many practically important cases take the following form for the physical components of the desired values (Ref 66):

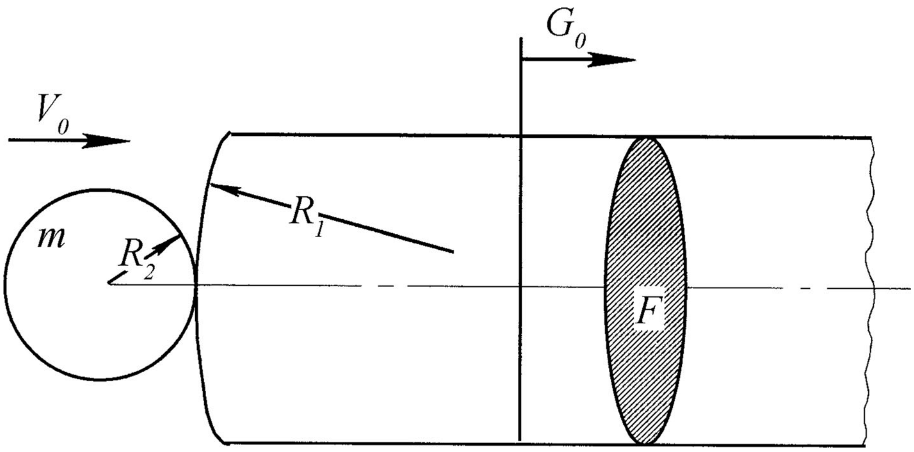

Fig. 1 Longitudinal impact of an elastic sphere with a bar (Ref 1)

$$G\left[\frac{\partial Z_{,(k)}}{\partial s}\right] = -\left[Z_{,(k+1)}\right] + \frac{\delta\left[Z_{,(k)}\right]}{\delta t} \qquad \text{(Eq 21)}$$

where s is the spatial coordinate along the straight ray, and the other two spatial coordinates are at a time the surface coordinates on the wave surface; thus, all three coordinate lines are mutually orthogonal, and $\delta/\delta t$ is the Thomas δ-derivative (Ref 67).

The recurrent relationships obtained as a result of such a procedure allow one to determine entering into the ray series (Eq 20) discontinuities in the derivative of the desired function with respect to time of any order.

2.4.1 Application of One-Term Ray Expansions Outside the Contact Domain. This approach was first used by Crook (Ref 63) when solving the problem on a longitudinal elastic impact of a sphere against an elastic bar (Fig. 1).

On the shock wave front and behind it up to the contact boundary, the dynamic condition of compatibility is fulfilled

$$\sigma = -\rho_0 G_0 V \qquad \text{(Eq 22)}$$

where σ is the stress, $G_0 = \sqrt{E_0/\rho_0}$ is the velocity of the shock wave, V is the velocity of the bar's particles along its axis, E_0 is the bar's modulus of elasticity, and ρ_0 is its density.

The contact force acting on the sphere takes the form

$$P = \rho_0 G_0 F V \qquad \text{(Eq 23)}$$

Considering equation (Eq 23) and the twice integrated equation of the sphere motion (Eq 8), from equation (Eq 11), we could finally find

$$\alpha = V_0 t - \frac{1}{m}\int_0^t dt_1 \int_0^{t_1} P(t_2)dt_2 - \frac{1}{\rho_0 G_0 F}\int_0^t P(t)dt \qquad \text{(Eq 24)}$$

If we substitute equation (Eq 12) into equation (Eq 24), then we obtain the functional equation with respect to the contact force P, while differentiating equation (Eq 24) twice with respect to time taking into account equation (Eq 2), we are led to equation (Eq 17), wherein $b = k(\rho_0 G_0 F)^{-1}$.

Wave approach has been extended to the problems of shock interaction of a sphere with a Timoshenko beam and with an Uflyand–Mindlin plate (Ref 1, 68). It is sometimes essential to take into account the extension of a middle surface in problems dealing with the shock interaction of impactors with thin-walled bodies, which are rather flexible elements of structures (Ref 1, 14, 15). Thus, in the later problem the motion of the impactor is described by (Eq 8), while the motion of the contact spot is governed by the following equation:

$$2\pi a N_r \frac{\partial w}{\partial r} + 2\pi a Q_r + P(t) = \rho h \pi a^2 \dot{W} \qquad \text{(Eq 25)}$$

where $W = \dot{w}$, w is the deflection, a is the radius of the contact spot defined by equation (Eq 6), Q_r is the transverse force, N_r is the force acting in the plate plane in the r-direction, and R is the sphere's radius (Fig. 2).

The dynamic behavior of a circular elastic isotropic plate of an Uflyand–Mindlin type in the polar coordinate system r, φ, z is described by the following set of equations:

$$\begin{gathered}\frac{1}{r}(M_r - M_\varphi) + \frac{\partial M_r}{\partial r} + Q_r = \rho I \dot{B}_r,\\ \dot{M}_r = D\left(\frac{\partial B_r}{\partial r} + \sigma\frac{B_r}{r}\right),\\ \dot{M}_\varphi = D\left(\frac{B_r}{r} + \sigma\frac{\partial B_r}{\partial r}\right)\\ \frac{\partial Q_r}{\partial r} + \frac{Q_r}{r} = \rho h \dot{W}, \quad \dot{Q}_r = K\mu h\left(\frac{\partial W}{\partial r} - B_r\right)\end{gathered} \qquad \text{(Eq 26)}$$

$$\frac{1}{r}(N_r - N_\varphi) + \frac{\partial N_r}{\partial r} = \rho h \dot{V}_r, \quad \dot{N}_r = E'h\left(\frac{\partial V_r}{\partial r} + \sigma\frac{V_r}{r}\right),$$
$$\dot{N}_\varphi = E'h\left(\frac{V_r}{r} + \sigma\frac{\partial V_r}{\partial r}\right)$$

where N_φ is the force acting in the plate plane in the φ-direction, M_r and M_φ are the bending moments, $V_r = \dot{u}_r$ is the displacement velocity along the radius, $B_r = \dot{\beta}_r$ is the angular velocity of the normal in the r-direction, $D = E'I$ is the plate cylindrical rigidity, $E' = E(1-\sigma^2)^{-1}$, $I = h^2/12$, and $K = \pi^2/12$ is the shear coefficient.

At the moment of impact, a shock wave (i.e., a surface of strong discontinuity) is generated in the plate, which is interpreted as a layer of the thickness δ, within which the desired function Z is changed from the magnitude Z^- to the

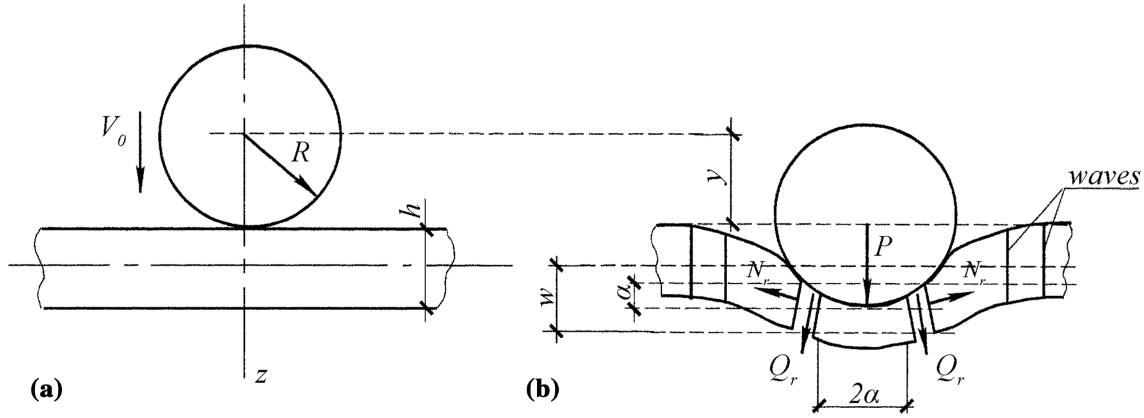

Fig. 2 Schematic diagram of an elastic impact upon an Uflyand–Mindlin plate considering the extension of its middle surface: (a) at the initial moment of impact $t = 0$, and (b) during the impact interaction $t > 0$

magnitude Z^+ remaining a continuous function. Then, integrating governing equations over the layer's thickness from $-\delta/2$ to $\delta/2$, tending δ to zero, and considering that according to (Eq 21) within the layer

$$\dot{Z} \approx -G\frac{\partial Z}{\partial r} \quad \text{(Eq 27)}$$

we obtain the dynamic conditions of compatibility

$$[Q_r] = -\rho h G[W], \quad -G[Q_r] = K\mu h[W] \quad \text{(Eq 28)}$$

$$[N_r] = -\rho h G[V_r], \quad -G[N_r] = E'h[V_r] \quad \text{(Eq 29)}$$

whence it follows that the velocities of the quasi-longitudinal wave $G^{(1)}$ and quasi-transverse wave $G^{(2)}$

$$G^{(1)} = \left(\frac{E'}{\rho}\right)^{1/2} \quad \text{(Eq 30)}$$

$$G^{(2)} = \left(\frac{K\mu}{\rho}\right)^{1/2} \quad \text{(Eq 31)}$$

and the discontinuities in the transverse force and longitudinal force

$$Q_r = -\rho h G^{(2)} W, \quad N_r = -\rho h G^{(1)} V_r \quad \text{(Eq 32)}$$

Now substituting (Eq 32) in (Eq 25) and considering that (Ref 1)

$$N_r = \rho G^{(1)2}\alpha(1-\sigma)/\sigma, \quad \text{(Eq 33)}$$

we arrive at the following equation for $A = \dot{\alpha}$

$$A\frac{dA}{d\alpha} + \frac{gb}{(e\alpha + g)^2}A = -\frac{k}{m}\alpha^{3/2} \quad \text{(Eq 34)}$$

where

$$b = \frac{k}{2\pi\rho R^{1/2}}, \quad e = \frac{G^{(1)2}}{G^{(2)2}}\frac{(1-\sigma)}{\sigma}, \quad g = G^{(2)}h$$

If we ignore the deformation of the middle surface (i.e., putting $e = 0$), then the analytical solution of (Eq 34) takes the form

$$A = V_0 - a_1\alpha + a_2\alpha^{5/2} + \sum_{n=2}^{\infty} a_{n+1}\alpha^{(2n+3)/2} \quad \text{(Eq 35)}$$

where

$$a_{n+1} = \prod_{m=2}^{n}\left(\frac{2m+1}{2m+3}\right)\left(\frac{a_1}{V_0}\right)^{n-1} a_2, \quad a_2 = -\frac{2}{5}\frac{k}{V_0 m},$$
$$a_1 = k\left(2\pi\rho h G^{(2)} R^{1/2}\right)^{-1}$$

Integrating yields an approximate solution for α as follows

$$\alpha \approx V_0 t - \frac{1}{2}a_1 V_0 t^2 + \frac{2}{7}a_2 V_0^{5/2} t^{7/2} \quad \text{(Eq 36)}$$

The coefficients a_1 and a_2 are defined by the two different processes being caused by the shock interaction: the coefficient a_1 is responsible for the dynamic processes arising in the plate during propagation of the surfaces of the discontinuity, but the coefficient a_2 answers for the quasi-static processes occurring at the local bearing of the material due to Hertzian theory.

This approach was extended for the analysis of the impact response of thin-walled beams of open profile without (Ref 1) and with due account (Ref 1, 15) for the middle surface extension. Nonlinear equation in terms of α was obtained similar to (Eq 34) with the corresponding analytical solution.

A normal impact of another type of the projectile in a form of a long cylindrical rod of the radius r_0 with a rounded end (Fig. 3) was also considered (Ref 1, 12). At the moment of impact, a longitudinal shock wave is generated in the rod, which propagates with the velocity G_0. The rod's length is such that the wave reflected from its free end has no time to reach the place of contact before the completion of the impact process. Behind the front of this wave, the dynamic condition of compatibility is fulfilled

$$\sigma^- = \rho_0 G_0 (V_0 - V^-) \quad \text{(Eq 37)}$$

where σ^- and V^- are the stress and velocity behind the wave front, which are valid up to the contact spot.

Suppose, for simplicity, that the radius of the contact spot is equal to r_0. Within the contact domain of the rod with the plate, the condition (Eq 37) takes the form

$$P = \pi r_0^2 \rho_0 G_0 (V_0 - W - \dot{\alpha}) \quad \text{(Eq 38)}$$

where the contact force is defined by (Eq 2).

The governing nonlinear equation was found (Ref 1, 15)

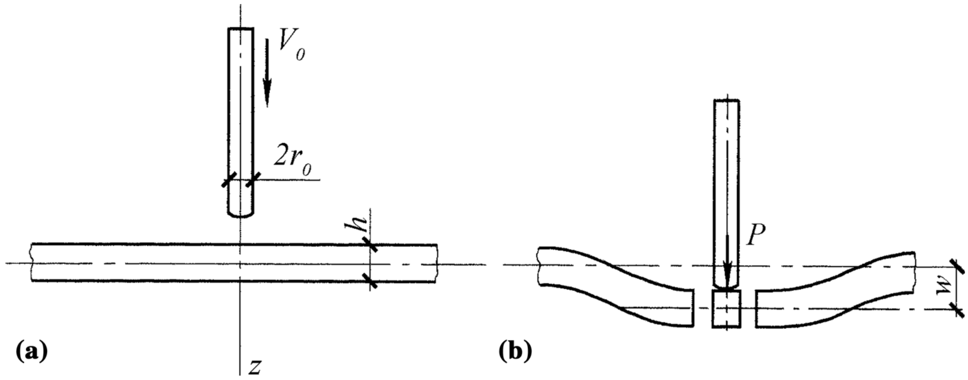

Fig. 3 Schematic diagram of an elastic cylindrical bar shock interaction with an Uflyand–Mindlin plate: (a) before impact, and (b) during the contact interaction

$$\ddot{\alpha}\alpha^{1/2} + \dot{\alpha}\left(\frac{3k}{2\rho_0 G_0 \pi r_0^2}\alpha + \frac{2G^{(2)}}{r_0^{1/2}}\right) + \frac{k}{\rho h \pi r_0}\alpha + \frac{2G^{(2)}k}{\rho_0 G_0 \pi r_0^{5/2}}\alpha^{3/2} = \frac{2G^{(2)}V_0}{r_0^{1/2}} \quad \text{(Eq 39)}$$

the solution of which could be constructed using the procedure described above.

The more intricate case when the motion of the rod is accompanied by its rotation was studied in Ref 1.

2.4.2 Application of Multiple-term Ray Expansions Outside the Contact Domain. Using multiple-term ray expansions (Eq 20) and conditions of compatibility (Eq 21) for equations of motion of the Uflyand–Mindlin plate (Eq 26), the following ray series were constructed for the velocity of the plate's deflection W and transverse force Q_r (Ref 1):

$$W \cong \sum_{\alpha=1}^{2}\sum_{k=0}^{\infty}\frac{1}{k!}\left[W_{,(k)}\right]^{(\alpha)}(y_\alpha)^k H(y_\alpha) \quad \text{(Eq 40)}$$

$$Q_r \cong K\mu h \sum_{\alpha=1}^{2}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\left[W_{,(k)}\right]^{(\alpha)}G^{(\alpha)-1} + \frac{\delta\left[W_{,(k-1)}\right]^{(\alpha)}}{\delta t}G^{(\alpha)-1} - \left[B_{r,(k-1)}\right]^{(\alpha)}\right)(y_\alpha)^k H(y_\alpha) \quad \text{(Eq 41)}$$

where $y_\alpha = t - (r-a)G^{(\alpha)-1}$, and the values $\left[W_{,(k-1)}\right]^{(\alpha)}$ and $\left[B_{r,(k-1)}\right]^{(\alpha)}$ and their δ-derivatives are calculated at $y_\alpha = 0$.

The following equations are fulfilled in the contact domain: the equation of motion of the contact spot (Eq 25), the equation of the sphere's motion (Eq 8), and the boundary condition within the contact domain

$$\left.\frac{\partial W}{\partial r}\right|_{r=a} = 0 \quad \text{(Eq 42)}$$

and the initial condition

$$W|_{t=0} = 0 \quad \text{(Eq 43)}$$

The solution of the enumerated equations, which form a system, is constructed with the help of the ray series (Eq 40) and (Eq 41) at $y_\alpha = t$, while the expression for the indentation α is chosen in the form

$$\alpha = V_0 t + \sum_{i=1}^{\infty}\alpha_i t^{i+1} \quad \text{(Eq 44)}$$

where α_i (i = 0,1,...) are yet unknown constants.

If we substitute relationships (Eq 40), (Eq 41), and (Eq 43) in the given set of equations and equate the coefficients at equal powers of t, then it is possible at each step to obtain three equations in three unknown constants, as the ray expansions and the function $\alpha(t)$ involve two constants and one constant, respectively. However, when choosing $\alpha(t)$ in the form of (Eq 44), the radius of the contact domain

$$a = R^{1/2}\alpha^{*1/2}\left[1 + \frac{1}{2}\alpha^{*-1}V_0(t-t^*) + \left(-\frac{1}{4}\alpha^{*-2}V_0^2 + \alpha^{*-1}\alpha_1\right)\times\frac{1}{2}(t-t^*)^2 + \left(\frac{9}{8}\alpha^{*-3}V_0^3 - \frac{3}{2}V_0\alpha_1\alpha^{*-2} + 3\alpha_2\alpha^{*-1}\right)\times\frac{1}{6}(t-t^*)^3 + \cdots\right] \quad \text{(Eq 45)}$$

since the shock waves go outside the contact domain only at $t > t^* = \frac{RV_0}{2G^{(1)2}}$ and propagate with the velocity $G^{(1)}$ (Ref 1, 11, 13).

The time dependence of the contact force for steel and aluminum plates of various thicknesses is shown in Fig. 4, from which it is seen that the data presented are in good agreement with the experimental results from Goldsmith (Ref 2).

2.5 Utilizing Non-classical Theories Considering Shear Deformations, Rotary Inertia and Changes in Microstructure for Describing Dynamic Behavior of Thin Bodies

One hundred years have passed since Stephen Timoshenko in his pioneer works (Ref 69, 70) generalized the Bernoulli–Euler beam model introducing into consideration two independent functions: the displacement of the center of gravity of beam's cross section and rotation of its cross section around the longitudinal central axis. Further Timoshenko approach was extended to plates and shells (Ref 71-74). The structural mechanics community has celebrated this event by a set of papers, among them the survey (Ref 75), wherein an interesting reader could find a lot of references in the field.

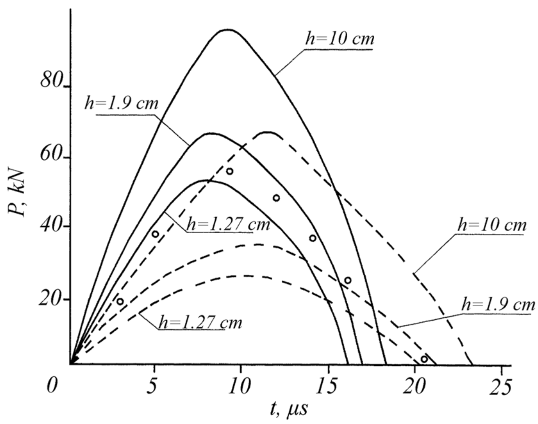

Fig. 4 Time dependence of the contact force during shock interaction of a steel sphere of R = 6.356 mm with initial velocity V_0 = 45.8 m/s with a plate of R_p= 300 mm: solid lines, steel plate (wave solution); dashed lines, aluminum plate (wave solution); circles, steel plate with h = 12.7 mm (solution cited in Goldsmith, Ref 2)

One of the limitations of the Timoshenko theory is that in the governing equations there exists a certain correction factor K (shear coefficient) which should be determined experimentally, and depending on the character of the experiment it can take on different magnitudes. Thus, $K^2 = 5/6$ is suggested in Ref 73, 74, and the values 2/3 and 8/9 can be found in Ref 70 and (Ref 71). Mindlin (Ref 72) suggested determining K^2 for plates reasoning from comparison of the elastic wave velocity on the basis of the accepted model with the corresponding velocity found by virtue of the 3D equations of the theory of elasticity. Then the magnitude of K^2 ranges from 0.76 to 0.91 with variation of the Poisson's ratio between 0 and 0.5. Sometimes the magnitude $K^2 = \pi^2/12$ (Ref 72, 76) is in use, which is obtained from comparison of the frequencies of the first antisymmetric mode of vibrations of a rectangular extended plate found by the strict theory and by the relationships taking shear and rotary inertia into account.

Kaneko (Ref 77) carried out an excellent review of all the various shear coefficients that have been tried before 1975. Later on, in 2001, Hutchinson (Ref 78) discussed shear coefficients for different geometries of the beam's cross section, in so doing shear coefficients depend on the Poisson's ratio and dimensions of the cross section. Zhilin (Ref 79) proposed the shear coefficient $K^2 = 5/(6 - \sigma)$ for a plate under bending deformation. The determination of transverse shear stiffnesses for sandwich and laminated plates could be found in Altenbach (Ref 80).

An alternative method of constructing the basic relationships of the theory of thin bodies is to expand the displacements or stresses into the series (power or functional) with respect to the normal coordinate and to hold a certain truncated series depending on the required accuracy and the character of a problem (Ref 81). Substituting these series into the boundary conditions on internal and external surfaces of a thin body results in differential equations, and substitution into the 3D equations of elasticity leads to recurrent symbolic relationships allowing one to determine all coefficients of the higher-order expansions. Under this approach, particular values entering by artificial means (as the shear factor in the Timoshenko model and its generalizations) are absent in the coefficients of equations. However, the cumbersome mathematical treatment and the severity of equations and boundary conditions are the essential drawbacks of this approach.

What is the main purpose to take shear deformations and rotary inertia into account in boundary-value dynamic problems? The matter is fact that thin-walled bodies in different engineering applications are often subjected to nonstationary transverse loads, resulting in the generation of transient waves of transverse shear which should be taken into consideration (Ref 7, 10, 20). Classical equations describing the dynamic behavior of thin bodies exclude the propagation of such waves. However transient waves propagate in the form of wave surfaces of strong or weak discontinuity. That is why it is quite natural for solving such problems to utilize the theory of discontinuities, which is based on the ray expansions and conditions of compatibility considering transverse deformations of thin bodies. In doing so, it is necessary to start from the three-dimensional equations describing the behavior of the material which the thin body is made of.

Such a novel approach, despite to the Timoshenko-type theories, does not involve new material constants such as the shear coefficient K, which is rather hard to be determined experimentally for many thin-walled structures, shells or thin-walled beams with open or closed profile as examples.

This approach has been developed by Professor Rossikhin and his research team since 2007, and during ten years an orderly theory describing the dynamic response of such thin bodies as elastic plates and shells (Ref 10), elastic beams (Ref 82), elastic spatially curved beams of open profile (Ref 16), thermoelastic thin-walled beams (Ref 83, 84) and thin-walled beams of open profile made of Cosserat pseudo-continuum (Ref 23, 85), has been created. The compact analytical expressions have been obtained for the velocities of the generalized displacements and the contact force, which easily could be calculated numerically and could be utilized by practical engineers for solving different dynamic contact interaction problems.

Recently the interest to the analysis of composite structures using Cosserat theory has been renewed (Ref 86) owing to the appearance of efficient techniques allowing one to reconstruct Cosserat moduli in materials using long waves (Ref 87) or to derive Cosserat moduli via homogenization of heterogeneous elastic materials (Ref 88). However it should be noted that in the majority of publications in the field, the authors have limited themselves by the consideration of static problems (Ref 86, 88) or harmonic wave propagation (Ref 87) (an interested reader could also find a lot of useful references within the above-mentioned articles Ref 23, 85-88).

2.5.1 Impact of an Elastic Rod with a Rounded End Against a Thin-Walled Beam of Arbitrary Open Profile. The problem of impact of a long elastic rod with a rounded end against a three-layered thin-walled spatially curved Cosserat beam of arbitrary open cross section (Fig. 5) has been considered recently by Rossikhin and Shitikova (Ref 23) starting from the three-dimensional equations of the Cosserat continuum (Ref 89), which on the wave surface of strong discontinuity have the form (Ref 23):

$$[\sigma_{ij,(k+1)}] = \lambda[v_{l,l(k)}]\delta_{ij} + \mu\big([v_{i,j(k)}] + [v_{j,i(k)}]\big) + \alpha\big([v_{i,j(k)}] - [v_{j,i(k)}] - 2 \in_{nji} [\omega_{n,(k)}]\big) \qquad \text{(Eq 46)}$$

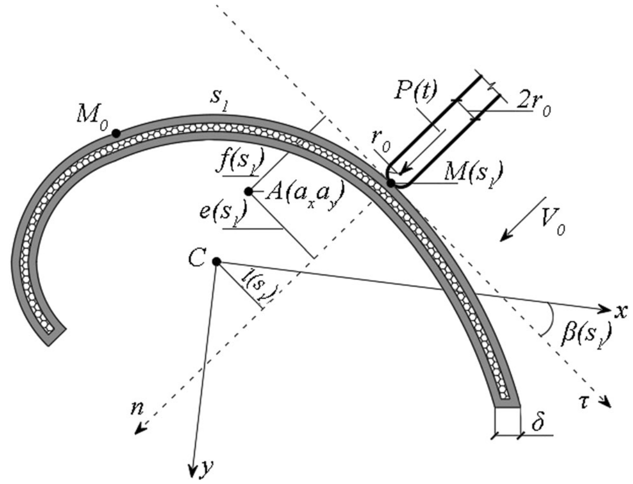

Fig. 5 Scheme of the impact interaction

$$[\mu_{ij,(k+1)}] = \beta[\omega_{l,l(k)}]\delta_{ij} + \gamma([\omega_{i,j(k)}] + [\omega_{j,i(k)}]) + \varepsilon([\omega_{i,j(k)}] - [\omega_{j,i(k)}]) \quad \text{(Eq 47)}$$

$$[\sigma_{ij,j(k)}] = \rho[v_{i,(k+1)}] \quad \text{(Eq 48)}$$

$$\in_{nji}[\sigma_{nj,(k)}] + [\mu_{ij,j(k)}] = J[\omega_{i,(k+1)}] \quad \text{(Eq 49)}$$

where σ_{ij} is the stress tensor, μ_{ij} is the moment stress tensor, $\in_{kij}$ are the Levi–Civita tensor components, u_i is the displacement vector, $v_i = \dot{u}_i$ is the velocity vector, an index after a coma labels a derivative with respect to the corresponding coordinate, ψ_i is the angular rotation vector, $\omega_i = \dot{\varphi}_i$ is the angular velocity vector, δ_{ij} is the Kronecker's symbol, J is the moment of inertia, $\lambda, \mu, \alpha, \beta, \gamma$ and ε are material constants, and x_i (i = 1,2,3) are Cartesian coordinates.

Now writing the compatibility conditions for the displacements on the front of the shock wave and considering that there are no cracks, i.e., $[u_i] = [\psi_i] = 0$, we find

$$[u_{i,j}] = -G^{-1}[v_i]\lambda_j + \left[\frac{\partial u_i \xi_j}{\partial \xi}\right] \quad \text{(Eq 50)}$$

$$[\psi_{i,j}] = -G^{-1}[\omega_i]\lambda_j + \left[\frac{\partial \psi_i \xi_j}{\partial \xi}\right] \quad \text{(Eq 51)}$$

The second terms in (Eq 50) and (Eq 51) are remained in order to take the transverse deformation of the shell into account in further treatment. Considering that on the free surface of the beam

$$[\sigma_{ij,(k)}]v_j = 0, \quad [\mu_{ij,(k)}]v_j = 0$$

using further the assumption

$$[\sigma_{ij}]\xi_i\xi_j = [\mu_{ij}]\xi_i\xi_j = 0$$

corresponding to the condition of nonpressing of beam's layers on each other during the wave front propagation, the velocities of transient waves have been found and classified as follows (Ref 23, 85):

velocity of the quasi-longitudinal wave

$$G_1 = \sqrt{\frac{E}{\rho}} \quad \text{(Eq 52)}$$

velocity of the quasi-shear wave

$$G_2 = \sqrt{\frac{\mu + \alpha}{\rho}} \quad \text{(Eq 53)}$$

velocity of the quasi-rotational micropolar wave

$$G_3 = \sqrt{\frac{e}{J}} \quad \text{(Eq 54)}$$

and velocity of the quasi-flexural micropolar wave

$$G_4 = \sqrt{\frac{\gamma + \varepsilon}{J}} \quad \text{(Eq 55)}$$

From the found four velocities of propagation of transient waves (surfaces of strong discontinuity), it is seen that (1) they depend only on material constants, and (2) only one micropolar modulus α, which governs the asymmetry of the stress tensor, influences the velocity of the quasi-shear wave G_2, while the Láme moduli λ and μ do not affect the velocities of Cosserat waves, i.e., the third G_3 and fourth G_4 waves which are generated due to micropolar rotations.

Behind the shock wave front propagating with the velocity $G_0 = \sqrt{E_0\rho_0^{-1}}$ along the elastic rod after its collision with the beam, the stress and velocities of the rod's particles could be represented by the following ray series (Ref 23):

$$\sigma^- = -\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k \sigma}{\partial t^k}\right]\left(t - \frac{n}{G_0}\right)^k \quad \text{(Eq 56)}$$

$$v^- = V_0 - \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k v}{\partial t^k}\right]\left(t - \frac{n}{G_0}\right)^k \quad \text{(Eq 57)}$$

which allow to define the contact stress $\sigma_{cont} = \sigma^-|_{n=0}$

$$\sigma_{cont} = \rho G_0(V_0 - v_n) \quad \text{(Eq 58)}$$

and the contact force

$$P = \pi r_0^2 \rho_0 G_0(V_0 - v_n) \quad \text{(Eq 59)}$$

But in this problem, the contact force could be also found from the Hertz contact law

$$P = k\vartheta^{3/2} \quad \text{(Eq 60)}$$

where k is the rigidity coefficient depending on the geometry of colliding bodies and their elastic constants (Eq 3).

Equations of motion of the contact domain (Fig. 5) could be written in the form (Ref 23)

$$2Q_{\lambda x} + P\sin\beta(s_1) = 0, \quad 2Q_{\lambda y} + P\cos\beta(s_1) = 0 \quad \text{(Eq 61)}$$

$$2M_A + Pe(s_1) = 0, \quad 2M_\lambda^m + M(P) = 0$$

where $M(P)$ is the moment which is generated within the contact layer due to the action of the contact force which induces the moment stresses in the core of the contact layer. Since the contact force P is considered to be small and $M(0) = 0$, then the moment $M(P)$ could be approximated as

$M(P) \approx M'(0)P$,, where a prime denotes the derivative with respect to P.

An approximate solution of the set of equations (Eq 59-61) has been found in Ref 23 as

$$\vartheta \approx V_0 t\left(1-\frac{2}{5}\,V_0^{1/2}\kappa t^{3/2}\right) \qquad \text{(Eq 62)}$$

where

$$\kappa = k\left\{\frac{1}{\pi r_0^2 \rho G_0} + \frac{1}{2}(\rho G_2)^{-1}\left[F^{-1} + \left(I_p^A\right)^1 e(s_1) l(s_1)\right] + \frac{1}{2}(JFG_3)^1 l(s_1) M'(0)\right\}, \qquad \text{(Eq 63)}$$

whence it follows the contact duration

$$t_{\text{cont}} = \left(\frac{5}{2}\,V_0^{-1/2}\kappa^{-1}\right)^{2/3} \qquad \text{(Eq 64)}$$

The maximal indentation $\vartheta_{\max}$ is reached at $\dot{\vartheta} = 0$

$$\vartheta_{\max} = \left(\frac{V_0}{\kappa}\right)^{2/3} \qquad \text{(Eq 65)}$$

corresponding to the maximum of the contact force

$$P_{\max} = k\vartheta_{\max}^{3/2} = kV_0\kappa^{-1} \qquad \text{(Eq 66)}$$

From relationship (Eq 63), it is seen that the influence of micropolar effect is described via the third term in the coefficient κ. Assuming that $M'(0) > 0$, it follows that the coefficient κ defined by (Eq 63) is larger in magnitude than that obtained in Rossikhin and Shitikova (Ref 20) for an elastic isotropic thin-walled beam of open profile impacted by an elastic rod with a rounded end.

Reference to (Eq 63-66) shows that the account for micropolar properties of the target material results in the decrease in magnitudes of all key parameters of the impact interaction, namely the indentation ϑ, its maximal value ϑ max, the contact duration t_{cont}, and the maximal contact force $P_{\max}$, as compared with those obtained in Ref 20 during the analysis of the impact response of the elastic isotropic thin-walled beam of open generic profile. In other words, the three-layered beams possessing micropolar properties are more impact resistance than the beams made of only one isotropic material.

3. Modeling the Impact Response of Thin Bodies Without Utilizing the Hertzian Contact Law

In this section, we consider some alternative approaches for solving dynamic problems of shock interaction, which are based on relationships different from the Hertzian contact law. These contact laws are modeled by linear or nonlinear springs or the combination of a spring with a damper.

As shown in the previous Sect. 2, the inclusion of the local deformation at the point of contact yields a nonlinear term in the integral equations (Eq 13) and (Eq 14) or the differential equation (Eq 17), and they have to be solved either numerically for bodies of a finite extent or by power series for bodies of an infinite extent. In certain cases, however, the frequency of the structural response is very low when compared to the high frequencies associated with local deformation, or the local deformation is much smaller as compared with the target deflection. Consequently, the local bearing could be neglected, and the resulting equation of motion during impact can be solved analytically, even for the cases when the motion of the target is described by nonlinear differential equation (Ref 90).

In some problems, local deformations at the place of contact are absent because of the nature of the problem or the problem statement. As an example is the impact by a thin rod with a plane end (Ref 1, 12, 24). Thus, the dynamic response of a prestressed transversely isotropic plate to impact by an elastic flat-end rod has been investigated in Rossikhin and Shitikova (Ref 18) using this approach. It has been shown that as the radial compression forces assess the critical magnitude, the velocity of the transient wave of transverse shear and its amplitude vanish. This leads to the fact that the portion of the impact energy that is expended on work of transverse forces is completely absorbed by the contact spot, resulting in the occurrence of the damage area within the contact region.

3.1 Modeling the Contact Interaction of Thin Bodies Via a Linear Elastic Spring

Linearization of the contact deformation is often useful for investigating the shock interaction of solids (Ref 1, 3). This can be done in different ways, and some examples are considered below.

3.1.1 Approach Based on One-Term Ray Expansions Outside the Contact Domain. It should be mentioned that this approach was first used by Conway and Lee (Ref 91) to analyze the impact between an indenter and a large elastic plate through a linear spring when investigating the mechanics of the printing process. The plate was sufficiently large to ignore reflections from its boundaries, so the velocity of the contact spot was proportional to the contact force; that is, equation (Eq 15) proposed by Zener (Ref 9) was valid. An elastic spring was located between the target and the indenter, so the contact force could be connected with the displacements of the indenter and the contact spot (the plate's displacement at the place of contact) by the following relationship:

$$P = E_1(\alpha - w) \qquad \text{(Eq 67)}$$

where E_1 is the spring rigidity, and α and w are the displacements of the spring's upper and lower ends, respectively.

The problem of the collision of a body with an Uflyand–Mindlin plate through a linear elastic cylindrical spring with the radius of r_0 (Fig. 6) was considered in Ref 1. In order to obtain in this case the differential equation for determining the contact force, it is sufficient to use equation (Eq 67) twice differentiated in time, the equation of impactor motion (Eq 8), the dynamic conditions of compatibility (32), and the equation of motion of the contact spot (25) at $a = r_0$.

Then the problem is reduced to a set of two equations:

$$\begin{aligned} M\ddot{w} &= -MB_1\dot{w} + E_1(\alpha - w) \\ m(\ddot{\alpha} + \ddot{w}) &= -E_1(\alpha - w) \end{aligned} \qquad \text{(Eq 68)}$$

The solution of (Eq 68) in the Laplace domain has the form (Ref 1)

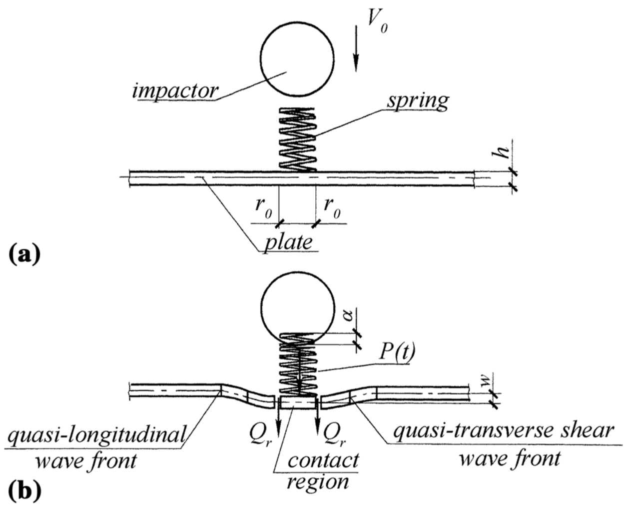

Fig. 6 A schematic diagram of impact interaction: (a) before impact; (b) after impact

$$\bar{w} = -\frac{p^2 + C_0}{p^2 - C_0}\bar{\alpha} + \frac{V_0}{p^2 - C_0} \quad \text{(Eq 69)}$$

$$\bar{\alpha} = V_0 \frac{p^2 + B_1 p + A_1}{p(p^3 + B_1 p^2 + C_1 p + C_0 B_1)} \quad \text{(Eq 70)}$$

where $M = \rho h \pi r_0^2$ is the mass of the contact domain, p is the Laplace variable, and

$$A_1 = \frac{E_1}{M}, \quad B_1 = \frac{2G^{(2)}}{r_0}, \quad C_1 = E_1\left(\frac{2}{M} + \frac{1}{m}\right) = 2A_1 + C_0, \quad C_0 = \frac{E_1}{m}$$

3.1.2 Approach Based on Multi-term Ray Expansions Outside Contact Domain. In order to analyze the impact response of an elastic isotropic Uflyand–Mindlin plate via the multiple-term ray expansion (Ref 1), it is needed to add the boundary condition (Eq 42), ray expansions (Eq 40) and (Eq 41) to the set of equations considered just above, as well as the expansion for α

$$\alpha = \sum_{i=1}^{6} \alpha_i t^i \quad \text{(Eq 71)}$$

Thus, we could find the time dependence of the contact force

$$\begin{aligned} P(t) = P^{\text{elast}}_{\text{isotr}} = E_1 V_0 \Bigg\{ & t - E_1\left(\frac{1}{m} + \frac{2}{M}\right)\frac{t^3}{6} + \frac{E_1(G^{(1)} + G^{(2)})}{Mr_0}\frac{t^4}{6} \\ & - E_1\left[\frac{(G^{(1)} + G^{(2)})^2}{Mr_0^2} - \frac{E_1}{6}\left(\frac{1}{m} + \frac{2}{M}\right)^2\right]\frac{t^5}{20} \\ & + \frac{E_1}{Mr_0}\left[\frac{(G^{(1)} + G^{(2)})\left[15(G^{(1)} + G^{(2)})^2 - G^{(1)}G^{(2)}\right]}{48r_0^2}\right. \\ & \left. - \frac{E_1}{3}(G^{(1)} + G^{(2)})\left(\frac{1}{m} + \frac{2}{M}\right) - \frac{G^{(2)2}(G^{(1)3} - G^{(2)3})}{h^2(G^{(1)2} - G^{(2)2})}\right]\frac{t^6}{30}\Bigg\} \end{aligned} \quad \text{(Eq 72)}$$

Note that because the duration of the shock interaction process is short, then it is reasonable to approximate the desired functions within the contact domain by the truncated power series with respect to time with an accuracy of t^6.

Rossikhin and Shitikova (Ref 1, 92) generalized this approach to investigate the low-velocity impact response of a circular prestressed elastic orthotropic plate possessing curvilinear anisotropy. The equations of motion of such a plate in the polar coordinate system taking into account transverse shear deformations and rotary inertia have the form

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} - \frac{E_\theta}{E_r}\frac{\varphi}{r^2} + b_r\left(\frac{\partial w}{\partial r} - \varphi\right) = \frac{\rho}{C_r}\ddot{\varphi} \quad \text{(Eq 73)}$$

$$\frac{\partial^2 w}{\partial r^2} - \frac{\partial \varphi}{\partial r} + \frac{1}{r}\left(\frac{\partial w}{\partial r} - \varphi\right) = \frac{\rho}{KG_{rz}}\ddot{w} + \frac{N}{hKG_{rz}}\Delta w \quad \text{(Eq 74)}$$

where N is the constant compression force acting in the radial direction, and $C_r = \frac{E_r}{1 - \sigma_r \sigma_\theta}$, $E_r \sigma_\theta = E_\theta \sigma_r$, $K = \frac{5}{6}$, $b_r = \frac{12KG_{rz}}{h^2 C_r}$, $\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$

Solving equations (Eq 73, 74), we can determine the contact force in the same manner as above for the isotropic plate free from preloading. In the case under consideration, the contact force with the accuracy of t^6 can be written by adding two additional terms to the relationship (Eq 72), where the last term at t^6 should be divided by f_r^2, where $f_r = 1 - N(hKG_{rz})^{-1}$, in order to take anisotropic features and preloading into account. As a result, we have

$$P(t) = P^{\text{preload}}_{\text{orthot}} = P^{\text{elast}}_{\text{isotr}} + \frac{E_1}{Mr_0}\left[\frac{G^{(1)3}G^{(2)}}{12r_0^2(G^{(1)} - G^{(2)})}\left(\frac{E_\theta}{E_r} - 1\right) + \frac{(1 - f_r)G^{(2)2}G^{(1)}}{f_r^2 h^2}\right]\frac{t^6}{30} \quad \text{(Eq 75)}$$

where $\rho G^{(1)2} = C_r$ and $\rho G^{(2)2} = KG_{rz}$.

Thus, a simple analytical relationship for the contact force (Eq 75) for any compressive force less than the critical force may be easily implemented in engineering practice required for the analysis of the low-velocity behavior of pre-compressed orthotropic plates.

At $N = N^{\text{crit}} = hKG_{rz}$ ($f_r = 0$, $G^{(2)} = 0$) is a critical state, which is most interesting, as in this case the plate occurs in the critical state, that is, only one wave is generated as a result of impact which further propagates with the velocity $G^{(1)}$, but the second wave turns out to be "locked" within the contact domain. The expression for the critical contact force can be written as

$$\begin{aligned} P^{\text{crit}}(t) = E_1 V_0 \Bigg\{ & t - E_1\left(\frac{1}{m} + \frac{2}{M}\right)\frac{t^3}{6} + \frac{E_1 G^{(1)}}{Mr_0}\frac{t^4}{6} \\ & - E_1\left[\frac{G^{(1)2}}{Mr_0^2} - \frac{E_1 M}{6}\left(\frac{1}{m} + \frac{2}{M}\right)^2\right]\frac{t^5}{20} \\ & + \frac{E_1 G^{(1)}}{Mr_0}\left[\frac{G^{(1)2}}{16r_0^2} - \frac{E_1}{15r_0^2}\left(\frac{1}{m} + \frac{2}{M}\right)\right]\frac{t^6}{30}\Bigg\} \end{aligned} \quad \text{(Eq 76)}$$

Equation 76 is important for designers of composite structures, as it can provide a basic understanding of the structural response in the critical state and how it is affected by different parameters, giving a foundation for the prediction of impact damage.

3.2 Modeling the Contact Interaction of Thin Bodies Via a Nonlinear Elastic Spring

The case of a nonlinear elastic spring the contact force defined as

$$P(t) = E_1(a - w) + E_2(a - w)^3 \quad \text{(Eq 77)}$$

where E_2 is the spring's nonlinear rigidity, was considered in Ref 1.

Using multiple-term ray expansions for the problem of impact of a sphere upon a nonlinear spring embedded into an elastic Kirchhoff–Love plate, the contact force was found

$$\begin{aligned} P(t) = P_{isot}^{nonlin} &= E_1 V_0 t + V_0 K t^3 + \frac{E_1^2 V_0 (G^{(1)} + G^{(2)})}{M r_0} \frac{t^4}{6} \\ &- E_1 V_0 \left\{ \frac{E_1 (G^{(1)} + G^{(2)})^2}{M r_0^2} + K \left(\frac{1}{m} + \frac{2}{M} \right) \right\} \frac{t^5}{20} \\ &+ \frac{E_1}{M r_0} \left\{ \frac{E_1 (G^{(1)} + G^{(2)}) \left[15 (G^{(1)} + G^{(2)})^2 - G^{(1)} G^{(2)} \right]}{48 r_0^2} \right. \\ &\left. - \frac{E_1 G^{(2)2} (G^{(1)3} - G^{(2)3})}{h^2 (G^{(1)2} - G^{(2)2})} + (G^{(1)} + G^{(2)}) \left[K - \frac{E_1^2}{6} \left(\frac{1}{m} + \frac{2}{M} \right) \right] \right\} \frac{t^6}{30} \end{aligned} \quad \text{(Eq 78)}$$

where $K = E_2 V_0^2 - \frac{E_1^2}{6} \left(\frac{1}{m} + \frac{2}{M} \right)$ is the generalized parameter of the nonlinear buffer. Such types of spring are used to protect overlapping plates supporting lift shafts.

Comparison of equations (Eq 76) and (Eq 78) shows that the nonlinear properties of the contact interaction influence the coefficients at t^3, t^5 and higher orders of t.

4. Conclusion

This survey article overviews the impact response of solids and structures within the framework of the wave theory of impact. It is dedicated to the bright memory of Professor Yury A. Rossikhin who has contributed a lot in the development of the wave theory of impact based on the theory of discontinuities and ray expansions, resulting in analytical solutions of intricate problems of impact interaction of solids possessing different features.

The wave theory of elastic impact has been extended by him to thermoelastic, viscoelastic and preloaded bodies. The main results in the field have been summarized in the state-of-the-art articles (Ref 1, 41, 59-61), as well as in monograph (Ref 20) and several entries in *Encyclopedia of Thermal Stresses* (Ref 8, 30) and *Encyclopedia of Continuum Mechanics* (Ref 10, 56-58) published by Springer.

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